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Scattering from singular potentials in quantum mechanics

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Abstract. In non-relativistic quantum mechanics, singular potentials in problems with spherical symmetry lead to a Schrödinger equation for stationary states with non-Fuchsian singularities both as $r \to 0$ and as $r \to \infty$. In the 1960s, an analytic approach was developed for the investigation of scattering from such potentials, with emphasis on the polydromy of the wavefunction in the r-variable. This paper extends those early results to an arbitrary number of spatial dimensions. The Hill-type equation which leads, in principle, to the evaluation of the polydromy parameter, is obtained from the Hill equation for a two-dimensional problem by means of a simple change of variables. The asymptotic forms of the wavefunction as $r \to 0$ and as $r \to \infty$ are also derived. The Darboux technique of intertwining operators is then applied to obtain an algorithm that makes it possible to solve the Schrödinger equation with a singular potential admitting a Laurent expansion, if the exact solution with even just one term is already known.

1. Introduction

One of the long-standing problems of non-relativistic quantum mechanics is the investigation of scattering from singular potentials, with efforts by many authors over several decades (see [1–15] and references therein). The main motivations can be described as follows [13].

(i) Repulsive singular potentials make it possible to obtain a fairly accurate description of the short-range part of the nucleon–nucleon interaction.

(ii) The (p,p) and (p, π) processes can be interpreted in terms of complex potentials r^{-n}, with $n > 2$.

(iii) Repulsive singular potentials reproduce also the interactions of nucleons with K-mesons, and α–α scattering processes.

(iv) The Lennard-Jones potential, proportional to r^{-12}, can be used to study interactions among the overlapping electron clouds of non-polar molecules.

(v) At a field-theoretical level, it appears quite remarkable that non-renormalizable field theories give rise to effective potentials in the Bethe–Salpeter equation which are singular [1, 2], whereas superrenormalizable and renormalizable field theories give rise to regular or transition-type effective potentials, respectively. There was therefore the hope that any new insight gained into the analysis of non-relativistic potential scattering in the singular case, could be eventually used to obtain a better understanding of quantum field theories for which perturbative renormalization fails (cf section 6).

(vi) In particular, one might then hope to be able to 'map' the analysis of quantum gravity based on the Einstein–Hilbert action (plus boundary terms), which is well known to be incompatible with the requirement of perturbative renormalizability [16], into a scattering problem in the singular case, for which the Schrödinger equation for stationary states:

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{(q-1)}{r}\frac{\mathrm{d}}{\mathrm{d}r} - \frac{l(l+q-2)}{r^2} + k^2\right]\psi(r) = V(r)\psi(r) \tag{1.1}$$

has non-Fuchsian singularities (see the appendix) both as $r \to 0$ and as $r \to \infty$. With our notation, q is the number of spatial dimensions, and $l(l+q-2)$ is obtained by studying the action of the Laplace–Beltrami operator on wavefunctions belonging to the tensor product [17]

$$L^2(\mathcal{R}_+, r^{q-1}\,\mathrm{d}r) \otimes L^2(S^{q-1}, \mathrm{d}\Omega).$$

Moreover, with a standard notation, one has (of course, the energy E is positive in a scattering problem)

$$k^2 \equiv \frac{2mE}{\hbar^2} \tag{1.2}$$

$$V(r) \equiv \frac{2m}{\hbar^2}U(r) \tag{1.3}$$

with $U(r)$ the potential term in the original form of the Schrödinger equation. From now on, it is $V(r)$ which will be referred to as *the potential*, following the convention in the literature.

Among the analytic results obtained so far in the investigation of potential scattering in the singular case, we find it appropriate to mention what follows.

(i) A constructive determination of the S-matrix, based on the polydromy properties of the wavefunction (see the appendix) and on the Hill equation for the polydromy parameter [6, 8].

(ii) Perturbative technique for the potential $V(r) \equiv g^2 r^{-4}$ in three dimensions, by re-expressing the radial Schrödinger equation as a modified Mathieu equation [10], with evaluation of S-matrix and Regge poles.

(iii) Generalized variable-phase approach, leading to a JWKB phase-shift formula [13].

(iv) Generalization of the JWKB method to arbitrary order, with rigorous error bounds [14].

In this paper, sections 2 and 3 apply the method of [6, 8] to the Schrödinger equation for stationary states in three or more spatial dimensions, proving that a simple but deep relation exists between the corresponding Hill equations in two and three or more spatial dimensions. Section 4 presents, for completeness, the JWKB analysis of the wavefunction, jointly with its limiting behaviour as $r \to 0$ and as $r \to \infty$. Section 5 studies the application of the intertwining operator technique to singular potential scattering. Results and open problems are described in section 6.

2. Schrödinger equation for stationary states

Following the remarks in the introduction, we first study the Schrödinger equation for stationary states in three spatial dimensions in a central potential:

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}}{\mathrm{d}r} - \frac{l(l+1)}{r^2} + k^2\right]\psi(r) = V(r)\psi(r). \tag{2.1}$$

What is crucial is the polydromy of the wavefunction in the r variable. Indeed, if the potential $V(r)$ is a single-valued function of r, one can find two independent solutions

$$\psi_1(r) = r^{\gamma}\chi_1(r) \tag{2.2}$$

$$\psi_2(r) = r^{-\gamma}\chi_2(r) \tag{2.3}$$

where χ_1 and χ_2 are single-valued functions of r, and γ is a parameter which can be determined from a transcendental equation (see below). The general solution of equation (2.1) is therefore of the form

$$\psi(r) = \alpha_1\psi_1(r) + \alpha_2\psi_2(r). \tag{2.4}$$

Remarkably, one can compute directly $\chi_1(r)$ and $\chi_2(r)$ and study their behaviour as $r \to 0$ and as $r \to \infty$ [6, 8]. For this purpose, the following Laurent expansions are used (the subscript for χ is omitted for simplicity):

$$W(r) \equiv r^2[V(r) - k^2] = \sum_{n=-\infty}^{\infty} w_n r^n \qquad r \in]0, \infty[\tag{2.5}$$

$$\chi(r) = \sum_{n=-\infty}^{\infty} c_n r^n \qquad r \in]0, \infty[. \tag{2.6}$$

These expansions hold because $V(r)$ is assumed to be an analytic function in the complex-r plane, with singularities only at infinity and at the origin [6, 8]. The Laurent series (2.5) and (2.6) are now inserted into equation (2.1), which is equivalent to the differential equation (cf [6, 8])

$$r^2\frac{\mathrm{d}^2\chi}{\mathrm{d}r^2} + 2(\gamma+1)r\frac{\mathrm{d}\chi}{\mathrm{d}r} + (\gamma(\gamma+1) - l(l+1))\chi = r^2[V(r) - k^2]\chi. \tag{2.7}$$

One thus finds the following infinite system of equations for the coefficients (cf [8]):

$$[(n+\gamma)(n+\gamma+1) - \bar{\lambda}^2]c_n = \sum_{m=-\infty}^{\infty} \bar{u}_{n-m}c_m \tag{2.8}$$

where

$$\bar{\lambda}^2 = l(l+1) + w_0 \tag{2.9}$$

$$\bar{u}_n = w_n - w_0\delta_{n,0}. \tag{2.10}$$

To solve the system (2.8) one first writes an equivalent system for which the determinant of the matrix of coefficients is well defined. Such a new system is obtained from (2.8) by dividing the nth equation by $(n+\gamma)(n+\gamma+1) - \overline{\lambda}^2$. The resulting matrix of coefficients has elements

$$H_{n,m} = \delta_{n,m} - \frac{\overline{u}_{n-m}}{[(n+\gamma)(n+\gamma+1) - \bar{\lambda}^2]} \tag{2.11}$$

where $\det(H)$ exists since the double series

$$\sum_{n,m} \frac{\overline{u}_{n-m}}{[(n+\gamma)(n+\gamma+1) - \bar{\lambda}^2]}$$

converges for all values of γ which do not correspond to zeros of the denominator. At this stage one can appreciate the substantial difference between regular and singular potentials. In the former case, u_n is non-vanishing only for positive n. In the singular case, however, the presence of negative powers in the Laurent series (2.5) gives γ as the solution of a transcendental equation, i.e. (the vanishing of $\det(H)$ being necessary and sufficient to find non-trivial solutions of the system (2.8))

$$F(\gamma) \equiv \det(H) = 0. \tag{2.12}$$

3. Equation for the γ parameter

To evaluate $F(\gamma)$, we point out that, on defining

$$\tilde{\gamma} \equiv \gamma + \tfrac{1}{2} \tag{3.1}$$

$$\tilde{\lambda}^2 \equiv \bar{\lambda}^2 + \tfrac{1}{4} = (l + \tfrac{1}{2})^2 + w_0 \tag{3.2}$$

one finds

$$(n+\gamma)(n+\gamma+1) - \bar{\lambda}^2 = (n+\tilde{\gamma})^2 - \tilde{\lambda}^2. \tag{3.3}$$

This simple but fundamental property makes it possible to perform the three-dimensional analysis by relying entirely on the investigation in two spatial dimensions, because $H_{n,m}$ now reads

$$H_{n,m} = \delta_{n,m} - \frac{\bar{u}_{n-m}}{[(n+\tilde{\gamma})^2 - \tilde{\lambda}^2]} \tag{3.4}$$

and hence, from the work in [6, 8], one knows that

$$F(\tilde{\gamma}) = 1 + [F(0) - 1]\frac{[\cot\pi(\tilde{\gamma}+\tilde{\lambda}) - \cot\pi(\tilde{\gamma}-\tilde{\lambda})]}{2\cot\pi\tilde{\lambda}} \tag{3.5}$$

where F is an even periodic function of $\tilde{\gamma}$, with unit period [8]. The equation

$$F(\tilde{\gamma}) = 0 \tag{3.6}$$

is, as we said in section 2, a transcendental equation. If a root, say x, is known, at least approximately, one can then evaluate the desired γ parameter from the definition (3.1) as

$$\gamma = x - \tfrac{1}{2}. \tag{3.7}$$

The ground is now ready for understanding the key features of singular potential scattering in an arbitrary number of spatial dimensions. For this purpose, we remark that, upon setting $\psi(r) = r^\gamma \chi(r)$, equation (1.1) leads to the following second-order equation for χ (cf (2.7)):

$$\left[r^2\frac{\mathrm{d}^2}{\mathrm{d}r^2} + (2\gamma+q-1)r\frac{\mathrm{d}}{\mathrm{d}r} + (\gamma^2 + (q-2)\gamma - l(l+q-2))\right]\chi(r) = W(r)\chi(r). \tag{3.8}$$

Thus, on defining (cf (3.1))

$$\tilde{\gamma} \equiv \gamma + \tfrac{1}{2}(q-2) \tag{3.9}$$

one can re-express equation (3.8) in the form

$$\left[r^2\frac{\mathrm{d}^2}{\mathrm{d}r^2} + (2\tilde{\gamma}+1)r\frac{\mathrm{d}}{\mathrm{d}r} + (\tilde{\gamma}^2 - \tfrac{1}{4}(q-2)^2 - l(l+q-2))\right]\chi(r) = W(r)\chi(r). \tag{3.10}$$

At this stage, the Laurent expansions (2.5) and (2.6) lead to an infinite system of equations for the coefficients c_n in the form (cf (2.8))

$$[(n+\tilde{\gamma})^2 - \tilde{\lambda}^2]c_n = \sum_{m=-\infty}^{\infty} \overline{u}_{n-m}c_m \tag{3.11}$$

where we have defined (cf (3.2))

$$\tilde{\lambda}^2 \equiv l(l+q-2) + \tfrac{1}{4}(q-2)^2 + w_0 = (l + \tfrac{1}{2}(q-2))^2 + w_0 \tag{3.12}$$

whereas the notation (2.10) remains unchanged. Thus, one can always perform the analysis in terms of the infinite matrix (3.4), provided that one defines $\tilde{\gamma}$ and $\tilde{\lambda}^2$ as in (3.9) and (3.12), respectively. The resulting Hill-type equation which leads, in principle, to the evaluation of the fractional part of the polydromy parameter γ, involves an even periodic function of $\gamma + \frac{1}{2}(q-2)$.

4. Asymptotic form of the solutions

Since one might be eventually interested in the S-matrix, it is quite important to study the limiting behaviour of stationary states as $r \to 0$ and as $r \to \infty$. In the former case, one can perform a JWKB analysis of equation (1.1), setting therein

$$\psi(r) = A(r)\mathrm{e}^{\frac{\mathrm{i}}{\hbar}S(r)}. \tag{4.1}$$

This leads to the equation (the prime denoting differentiation with respect to r)

$$\Bigg[(2m(E-U(r)) - S'^2)A + \mathrm{i}\hbar\left(2A'S' + AS'' + \frac{(q-1)}{r}AS'\right) + \hbar^2\left(A'' + \frac{(q-1)}{r}A' - \frac{l(l+q-2)}{r^2}A\right)\Bigg] = 0. \tag{4.2}$$

If, in a first approximation, the term on the second line ofequation (4.2) is neglected, one finds the equations

$$S'^2 = 2m(E-U(r)) \tag{4.3}$$

$$\frac{\mathrm{d}}{\mathrm{d}r}(A^2S') + \frac{(q-1)}{r}A^2S' = 0 \tag{4.4}$$

which imply

$$S' = \pm\sqrt{2m(E-U(r))} \tag{4.5}$$

$$A^2S' = \text{constant} \times r^{-(q-1)} \tag{4.6}$$

and hence, for some constant β,

$$A(r) = \beta r^{-\frac{(q-1)}{2}}(2m(E-U(r)))^{-\frac{1}{4}}. \tag{4.7}$$

To second order in $\hbar$, one has to consider the second line of equation (4.2). On taking the prefactor $A(r)$ in the form (4.7), one has then to evaluate the phase $S(r)$ from the equation

$$S'(r) = \pm\sqrt{2m(E-U(r)) + \hbar^2 f_{lq}(A(r))} \tag{4.8}$$

where

$$\begin{aligned} f_{lq}(A(r)) &\equiv \frac{A''}{A} + \frac{(q-1)}{r}\frac{A'}{A} - \frac{l(l+q-2)}{r^2} \\ &= -[\tfrac{1}{4}(q^2-4q+3) + l(l+q-2)]r^{-2} \\ &\quad + \tfrac{m}{2}U''(2m(E-U(r)))^{-1} + \tfrac{5}{4}m^2U'^2(2m(E-U(r)))^{-2}. \end{aligned} \tag{4.9}$$

Only an approximate calculation of the square root in equation (4.8) is possible, if

$$\rho \equiv \frac{\hbar^2 f_{lq}(A(r))}{2m(E-U(r))} \ll 1$$

by expanding $\sqrt{1+\rho}$ in powers of ρ, but this does not improve substantially the understanding of the behaviour of the wavefunction as $r \to 0$ for a fixed value of k (see below), and hence we do not present further calculations along these lines. One should bear in mind, however, that the JWKB expansion has an asymptotic nature, and rigorous error bounds can be obtained [14].

In particular, in the physically more relevant case of three spatial dimensions, equation (4.7) leads to (see (1.2) and (1.3))

$$A(r) = \tilde{\beta}r^{-1}(k^2 - V(r))^{-\frac{1}{4}}. \tag{4.10}$$

When $r \to 0$, $V(r)$ is much larger than k^2 *for a fixed value* of k, and hence the JWKB ansatz (4.1) leads to (hereafter $\hbar = 1$)

$$\psi_{I,II} \sim B_{I,II} r^{-1} (V(r))^{-\frac{1}{4}} \exp \int_r^{r_0} \sqrt{V(y)}\, \mathrm{d}y \tag{4.11}$$

for some parameters $B_{I,II}$ depending on γ and l (cf [6, 8, 18]). Of course, the JWKB solution for *all values* of k which results from (4.5) and (4.10) is, instead,

$$\psi_{I,II} \sim \tilde{\beta}_{I,II} r^{-1} (k^2 - V(r))^{-\frac{1}{4}} \exp \left[\mathrm{i} \int_{r_0}^{r} \sqrt{k^2 - V(y)}\, \mathrm{d}y \right]. \tag{4.12}$$

Moreover, as $r \to \infty$, one has the familiar asymptotic behaviour

$$\psi_{I,II} \sim A^{+}_{I,II} r^{-1} \exp \left\{ \mathrm{i} \left[kr - l\frac{\pi}{2} \right] \right\} + A^{-}_{I,II} r^{-1} \exp \left\{ -\mathrm{i} \left[kr - l\frac{\pi}{2} \right] \right\}. \tag{4.13}$$

The S-matrix is given by the formula [6, 8]

$$S = \frac{(A^{+}_{I} B_{II} - A^{+}_{II} B_{I})}{(A^{-}_{I} B_{II} - A^{-}_{II} B_{I})} \tag{4.14}$$

where the A and B parameters are the ones occurring in the asymptotic expansions (4.11) and (4.13), and can be obtained by means of the saddle-point method [6, 8].

5. Intertwining operators for singular potentials

Since exact solutions of singular scattering problems in terms of special functions are known in a few cases only, it appears quite important to look for a technique that makes it possible to generate solutions for complicated problems, relying on what is known in simpler cases. For this purpose, we here consider the Darboux method of intertwining operators [19–22].

The aim of the Darboux method is to generate families of isospectral Hamiltonians. It relies on a theorem which, in modern language, can be stated as follows [23]. Let ψ be the general solution of the Schrödinger equation

$$H\psi(x) \equiv \left[-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x) \right] \psi(x) = E\psi(x). \tag{5.1}$$

If φ is a particular solution of (5.1) corresponding to an energy eigenvalue $\varepsilon \neq E$, then

$$\tilde{\psi} = \frac{1}{\varphi} \left(\psi \frac{\mathrm{d}\varphi}{\mathrm{d}x} - \frac{\mathrm{d}\psi}{\mathrm{d}x} \varphi \right) \tag{5.2}$$

is the general solution of the Schrödinger equation

$$\tilde{H}\tilde{\psi}(x) = E\tilde{\psi}(x) \tag{5.3}$$

where

$$\tilde{H} \equiv -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \tilde{V}(x) \tag{5.4}$$

$$\tilde{V}(x) \equiv V(x) - 2\frac{\mathrm{d}^2}{\mathrm{d}x^2} \log \varphi(x). \tag{5.5}$$

In other words, if two Hamiltonian operators, say H_A and H_B, are given, one looks for a differential operator, say D, such that [24]

$$H_B D = D H_A. \tag{5.6}$$

It is then possible to relate the eigenfunctions of H_A and H_B by using the action of D (see below). Here, we focus on one-dimensional problems, with

$$H_A = H_0 \equiv -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_0(x) \tag{5.7}$$

$$H_B = H_1 \equiv -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_1(x) \tag{5.8}$$

$$D \equiv \frac{\mathrm{d}}{\mathrm{d}x} + G(x) \tag{5.9}$$

where V_0 and V_1 are the 'potential' functions, and G is another function, whose form is determined by imposing the condition (5.6). This reads, explicitly,

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_1\right)\left(\frac{\mathrm{d}}{\mathrm{d}x} + G\right) f = \left(\frac{\mathrm{d}}{\mathrm{d}x} + G\right)\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_0\right) f \tag{5.10}$$

for all functions f which are at least of class C^3. On imposing equation (5.10), one finds exact cancellation of the terms $-\frac{\mathrm{d}^3 f}{\mathrm{d}x^3}$ and $-G\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}$, since they occur on both sides with the same sign. Hence one deals with the equation

$$\left[(-2G' + V_1 - V_0)\frac{\mathrm{d}}{\mathrm{d}x} + (-G'' - V_0' + (V_1 - V_0)G)\right] f = 0 \tag{5.11}$$

which implies

$$2G' = V_1 - V_0 \tag{5.12}$$

$$-G'' - V_0' + (V_1 - V_0)G = 0 \tag{5.13a}$$

by virtue of the arbitrariness of f. It is now possible to use equation (5.12) to express equation (5.13a) in the form

$$\frac{\mathrm{d}}{\mathrm{d}x}(-G' + G^2) = \frac{\mathrm{d}}{\mathrm{d}x} V_0 \tag{5.13b}$$

which is solved by

$$G^2 - G' = V_0 + C \tag{5.14}$$

for some constant C. Equation (5.14) is known as the Riccati equation. Its nonlinear nature makes it desirable to develop an algorithm to relate it, instead, to the solution of a linear problem. This is indeed achieved by considering the function φ such that

$$G = -\frac{\mathrm{d}}{\mathrm{d}x} \log \varphi. \tag{5.15}$$

The equations (5.12) and (5.15) are, of course, in complete agreement with the result (5.5), with $\tilde{V}$ replaced by V_1, and V replaced by V_0. One then finds, by virtue of (5.14) and (5.15), that φ obeys the linear second-order equation

$$H_0 \varphi = -C\varphi. \tag{5.16}$$

This is a simple but deep result: one first has to find the eigenfunctions of H_0, say φ, belonging to the eigenvalue $-C$. Once this is achieved, the desired function G is obtained from (5.15), and hence the intertwining operator is

$$D = \frac{\mathrm{d}}{\mathrm{d}x} - \frac{\varphi'}{\varphi}. \tag{5.17}$$

In the applications, it is also convenient to use equation (5.12) to express equation (5.14) in the form

$$G^2 + G' = V_1 + C. \tag{5.18}$$

If one studies equation (1.1) or, in particular, equation (2.1), the standard definition in three dimensions

$$\psi(r) \equiv \frac{y(r)}{r} \tag{5.19}$$

leads to a second-order differential operator acting on y which is of the form (5.7) or (5.8). However, the choice of a suitable intertwining operator, aimed at relating operators H_A and H_B whose potential terms differ in a somehow substantial way, is a non-trivial task. For example, if one considers

$$V_1(r) \equiv \frac{A}{r^4} + \frac{B}{r^3} \tag{5.20}$$

equation (5.18) may be then satisfied by (cf (5.9))

$$G(r) = \frac{k}{r^2} \tag{5.21}$$

provided that $A = k^2$, $B = -2k$ and $C = 0$. However, the resulting potential $V_0(r)$ is found to be, from equation (5.12),

$$V_0(r) = \frac{k^2}{r^4} + \frac{2k}{r^3} \tag{5.22}$$

so that the intertwining operator ends up by relating operators H_A and H_B whose potential terms have precisely the same functional form. A scheme of broader validity, however, is obtained by looking for $V_1(r)$ and $G(r)$ in the form of (Laurent) series, i.e.

$$V_1(r) = \sum_{n=-\infty}^{\infty} a_n r^n \tag{5.23}$$

$$G(r) = \sum_{p=-\infty}^{\infty} b_p r^p. \tag{5.24}$$

The insertion of (5.23) and (5.24) into equation (5.18) leads to the infinite system

$$(n+1)b_{n+1} + \sum_{p=-\infty}^{\infty} b_p b_{n-p} = a_n + C\delta_{n,0} \tag{5.25}$$

that should be solved, in principle, for b_n, for all n. One then finds, from equation (5.12), a Laurent series for V_0 as well, i.e.

$$V_0(r) = \sum_{n=-\infty}^{\infty} f_n r^n \tag{5.26}$$

where

$$f_n = -(n+1)b_{n+1} + \sum_{p=-\infty}^{\infty} b_p b_{n-p} - C\delta_{n,0}. \tag{5.27}$$

For example, if one takes $V_1(r) \equiv g^2 r^{-4}$, one has

$$a_n = g^2 \delta_{n,-4} \tag{5.28}$$

and hence one deals with the infinite system

$$(n+1)b_{n+1} + \sum_{p=-\infty}^{\infty} b_p b_{n-p} = g^2 \delta_{n,-4} + C\delta_{n,0}. \tag{5.29}$$

This is a nonlinear algebraic system for which it does not seem possible to obtain a solution such that only a few b_p coefficients are non-vanishing. For example, if one tries to get $b_p = 0$ unless $p = -3, -2, -1$, one finds, on setting $n = -6, -5, -4, -3, -2, 0$ in (5.29) the six equations

$$(b_{-3})^2 = 0 \tag{5.30}$$

$$2b_{-3}b_{-2} = 0 \tag{5.31}$$

$$-3b_{-3} + 2b_{-3}b_{-1} + (b_{-2})^2 = g^2 \tag{5.32}$$

$$2(b_{-1} - 1)b_{-2} = 0 \tag{5.33}$$

$$(b_{-1} - 1)b_{-1} = 0 \tag{5.34}$$

$$0 = C \tag{5.35}$$

whereas $n = -1$ leads to a trivial identity. Now equations (5.30) and (5.31) imply that $b_{-3} = b_{-2} = 0$, and hence $g^2 = 0$ from (5.32), which is incompatible with our assumptions. The remaining equations (5.33)–(5.35) allow for $b_{-1} = 1$, further to $b_{-1} = 0$, but with $C = 0$.

However, the implications remain of high interest: to find non-trivial solutions with $g^2 \neq 0$ and $C \neq 0$ one needs a large number of b_p coefficients (maybe infinitely many), including those with $p > 0$. This still means that one has the opportunity to solve the Schrödinger equation with a complicated singular potential, starting from what one knows when the potential equals $g^2 r^{-4}$ [8]. For this purpose, on denoting again by φ the eigenfunction of H_0 belonging to the eigenvalue $-C$, and by $\chi \equiv D\varphi$ the eigenfunction of H_1 belonging to the same eigenvalue, we notice that the desired φ can be written in the form

$$\varphi(r) = \int_0^{\infty} K(r, r')\chi(r')\, \mathrm{d}r' \tag{5.36}$$

where $K(r, r')$ denotes the Green kernel of the intertwining operator $D \equiv \frac{\mathrm{d}}{\mathrm{d}r} + G(r)$. We need such an integral formula because we have chosen, in our particular example, the form of the potential term $V_1(r)$ in the Hamiltonian H_1, for which the scattering states are already known in the literature [8]. The unknown are instead the scattering states resulting from the Hamiltonian operator with potential term equal to $V_0(r)$ (see (5.26)).

6. Concluding remarks

Our paper has studied some aspects of scattering from singular potentials in quantum mechanics. Its contributions are as follows.

(i) The technique of Fubini and Stroffolini [6, 8], with emphasis on the polydromy properties of the wavefunction, has been applied to an arbitrary number of spatial dimensions, say q, when the potential admits a Laurent series expansion. The equation obeyed by the polydromy parameter, γ, involves a function which is an even periodic function of $\gamma + \frac{1}{2}(q-2)$. Interestingly, one can rely entirely on the analysis performed in [6, 8], provided that one considers the parameters defined in (3.9) and (3.12) (strictly, the authors of [6, 8] start from three dimensions, but use a transformation [9] leading to an equation formally analogous to the radial part of the stationary Schrödinger equation in two dimensions).

(ii) The Darboux technique of intertwining operators has been applied to relate the singular potential terms in the Schrödinger equation for stationary states. The algorithm resulting from equations (5.23)–(5.27) leads, in particular, to the nonlinear algebraic system (5.29) if the potential V_1 is taken to be $g^2 r^{-4}$.

Ultimately, one might want to use these properties to study quantum field theories which are not perturbatively renormalizable, according to the original motivations for this research field [1, 2, 9]. For this purpose, it seems crucial, to us, to consider the quantum gravity problem, focusing (at least) on the following questions.

(i) What is the counterpart, in quantum gravity, of the Bethe–Salpeter equation containing effective potentials of the singular type? As is well known, this equation arises in the course of studying the quantum theory of relativistic bound states, and unfortunately a simple extension of the Schrödinger equation is not available [25]. Even on neglecting curvature effects due to gravitational fields, one then faces retardation effects which lead to an extra relative time variable in the problem [25]. An alternative description uses a mediating field, whose quantum properties, however, cannot be ignored [25]. When a quantum theory of gravity is considered in a spacetime approach [26], one may expect to be able to use the (formal) theory of the effective action, with a corresponding set of integro-differential equations. These should be solved, in principle, by using the functional calculus. But even if one were able to achieve so much, and hence derive an effective potential which is a gravitational counterpart of the potential normally used to reduce the number of degrees of freedom of relativistic bound-state problems, the problem of giving a proper interpretation of such potentials would remain, since they seriously affect the exact theory and may introduce fictitious singularities (see p 493 of [25]).

(ii) What kind of results can be then 'imported' on mapping quantum gravity into a scattering problem from singular potentials (e.g. the asymptotic behaviour of the phase shift [9, 13], the exact or approximate solutions derived with some particular choices of singular potentials [1–8, 10, 14], or the existence theorem for the wave operators [12])?

(iii) How fundamental is the Darboux method of intertwining operators [19–22] proposed in section 5? Since the variable phase approach to potential scattering also relies on a Riccati-type equation [7], such a question appears to be non-trivial.

The above issues seem to point out that new perspectives are in sight in the analysis of potential scattering for a wide class of singular potentials, with possible implications for a long-standing problem, i.e. the key features of a quantum theory of the gravitational field (see [16] and references therein). Hence we hope that this paper, although devoted to some technical issues, may lead to a thorough investigation of quantum gravity from a point of view well-grounded in the general framework of modern high energy physics (cf [27]).

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Appendix

To be self-contained, let us describe what is meant by Fuchsian singularities of second-order differential equations. A theorem due to the German mathematician Immanuel Lazarus Fuchs states that a necessary and sufficient condition for the linear equation

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}x^2} + p_1(x)\frac{\mathrm{d}}{\mathrm{d}x} + p_2(x)\right] y(x) = 0 \tag{A1}$$

to admit a fundamental system of integrals, say $y_1(x)$ and $y_2(x)$, which, in the neighbourhood of the singular point x_0, can be expressed as (with $\varphi_1, \varphi_2, \psi$ analytic functions in the neighbourhood of x_0, for some constants $r_1, r_2, \tilde{r}_1, A$ and $(r_1 - r_2) \notin Z$)

$$y_1(x) = (x - x_0)^{r_1}\varphi_1(x) \tag{A2a}$$

$$y_2(x) = (x - x_0)^{r_2}\varphi_2(x) \tag{A2b}$$

or

$$y_1(x) = (x - x_0)^{\tilde{r}_1}\varphi_1(x) \tag{A3a}$$

$$y_2(x) = y_1(x)[A \log(x - x_0) + \psi(x)] \tag{A3b}$$

is that p_1 and p_2 should have poles of order $\leqslant 1$ and $\leqslant 2$, respectively, at the singular point x_0. One then says that x_0 is a *Fuchsian singularity* for equation (A1).

To study the point at infinity, one defines

$$\xi \equiv \frac{1}{x} \tag{A4}$$

which leads to the equation (cf (A1))

$$\left\{\frac{\mathrm{d}^2}{\mathrm{d}\xi^2} + \left[\frac{2}{\xi} - \frac{1}{\xi^2}p_1\left(\frac{1}{\xi}\right)\right]\frac{\mathrm{d}}{\mathrm{d}\xi} + \frac{1}{\xi^4}p_2\left(\frac{1}{\xi}\right)\right\} y(\xi) = 0. \tag{A5}$$

In the analysis of equation (A5) as $\xi \to 0$, which corresponds to the point at infinity of (A1), one can thus use again the Fuchs theorem, which implies that $p_1(\frac{1}{\xi})$ and $p_2(\frac{1}{\xi})$ should have zeros of degree $\geqslant 1$ and $\geqslant 2$, respectively, at $\xi = 0$. If this condition is fulfilled, equation (A1) is said to be *Fuchsian at infinity*.

When all singular points are Fuchsian, the corresponding differential equation is said to be *totally Fuchsian*. The *non-Fuchsian singularities* are, by contrast, singular points of (A1) for which the above conditions on poles and zeros of the functions p_1 and p_2 are not fulfilled.

In this paper, the word *polydromy* refers to the well known property of some functions of complex variable being multivalued functions of the independent variable. For example, if z is complex, its logarithm is given by the formula

$$\log(z) = \log|z| + \mathrm{i}\arg(z). \tag{A6}$$

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